

# Canonical Connection on a Class of Riemannian Almost Product Manifolds \*

Dimitar Mekerov, Dobrinka Gribacheva

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**Abstract.** The canonical connection on a Riemannian almost product manifolds is an analogue to the Hermitian connection on an almost Hermitian manifold. In this paper we consider the canonical connection on a class of Riemannian almost product manifolds with nonintegrable almost product structure.

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**Keywords.** Riemannian almost product manifold, Riemannian metric, nonintegrable structure, almost product structure, canonical connection

## 1 Introduction

On an Hermitian manifold  $(M, J, g)$  there exists an unique linear connection  $D$  with a torsion tensor  $T$  such that  $DJ = Dg = 0$  and  $T(x, Jy) = T(Jx, y)$  for all vector fields  $x, y$  on  $M$ . This is the Hermitian connection of the manifold ([4], [5], [1]). The group of the conformal transformations of the metric  $g$  generates the conformal group of the transformations of  $D$ . Analogously to the Hermitian connection on an almost Hermitian manifold, V. Mihova in [7] find the canonical connection on a Riemannian almost product manifold.

The systematic development of the theory of Riemannian almost product manifolds was started by K. Yano [10]. In [8] A. M. Naveira gives a classification of these manifolds with respect to the covariant differentiation of the almost product structure. Having in mind the results in [8], M. Staikova and K. Gribachev give in [9] a classification of the Riemannian almost product manifolds with zero trace of the almost product structure.

In the present work we consider the canonical connection on the manifolds of the class  $\mathcal{W}_3$  from the classification in [9].

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## 2 Preliminaries

Let  $(M, P, g)$  be a *Riemannian almost product manifold*, i.e. a differentiable manifold  $M$  with a tensor field  $P$  of type  $(1, 1)$  and a Riemannian metric  $g$  such that

$$P^2x = x, \quad g(Px, Py) = g(x, y) \quad (1)$$

for arbitrary  $x, y$  of the algebra  $\mathfrak{X}(M)$  of the smooth vector fields on  $M$ . Obviously  $g(Px, y) = g(x, Py)$ .

Further  $x, y, z, w$  will stand for arbitrary elements of  $\mathfrak{X}(M)$ .

In this work we consider Riemannian almost product manifolds with  $\text{tr}P = 0$ . In this case  $(M, P, g)$  is an even-dimensional manifold.

If  $\dim M = 2n$  then the *associated metric*  $\tilde{g}$  of  $g$ , determined by  $\tilde{g}(x, y) = g(x, Py)$ , is an indefinite metric of signature  $(n, n)$ . Since  $\tilde{g}(Px, Py) = \tilde{g}(x, y)$ , the manifold  $(M, P, \tilde{g})$  is a *pseudo-Riemannian almost product manifold*.

The classification in [9] of Riemannian almost product manifolds is made with respect to the tensor field  $F$  of type  $(0, 3)$ , defined by

$$F(x, y, z) = g((\nabla_x P)y, z), \quad (2)$$

where  $\nabla$  is the Levi-Civita connection of  $g$ . The tensor  $F$  has the following properties:

$$F(x, y, z) = F(x, z, y) = -F(x, Py, Pz), \quad F(x, y, Pz) = -F(x, Py, z). \quad (3)$$

The basic classes of the classification in [9] are  $\mathcal{W}_1$ ,  $\mathcal{W}_2$  and  $\mathcal{W}_3$ . Their intersection is the class  $\mathcal{W}_0$  of the *Riemannian  $P$ -manifolds*, determined by the condition  $F(x, y, z) = 0$  or equivalently  $\nabla P = 0$ . In the classification there are include the classes  $\mathcal{W}_1 \oplus \mathcal{W}_2$ ,  $\mathcal{W}_1 \oplus \mathcal{W}_3$ ,  $\mathcal{W}_2 \oplus \mathcal{W}_3$  and the class  $\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3$  of all Riemannian almost product manifolds.

In the present work we consider manifolds from the class  $\mathcal{W}_3$ . This class is determined by the condition

$$\mathfrak{S}_{x, y, z} F(x, y, z) = 0, \quad (4)$$

where  $\mathfrak{S}_{x, y, z}$  is the cyclic sum by  $x, y, z$ . This is the only class of the basic classes  $\mathcal{W}_1$ ,  $\mathcal{W}_2$  and  $\mathcal{W}_3$ , where each manifold (which is not Riemannian  $P$ -manifold) has a nonintegrable almost product structure  $P$ . This means that in  $\mathcal{W}_3$  the Nijenhuis tensor  $N$ , determined by

$$N(x, y) = (\nabla_x P)Py - (\nabla_{Px}P)y + (\nabla_y P)Px - (\nabla_{Py}P)x, \quad (5)$$

is non-zero.

In [9] it is introduced an *associated tensor*  $N^*$  by

$$N^*(x, y) = (\nabla_x P) Py + (\nabla_{Px} P) y + (\nabla_y P) Px + (\nabla_{Py} P) x. \quad (6)$$

It is proved that the condition (4) is equivalent to  $N^*(x, y) = 0$ .

Further, manifolds of the class  $\mathcal{W}_3$  we call *Riemannian  $\mathcal{W}_3$ -manifolds*.

As it is known the curvature tensor field  $R$  of a Riemannian manifold with metric  $g$  is determined by  $R(x, y)z = \nabla_x \nabla_y z - \nabla_y \nabla_x z - \nabla_{[x, y]} z$  and the corresponding tensor field of type (0, 4) is defined as follows  $R(x, y, z, w) = g(R(x, y)z, w)$ .

Let  $(M, P, g)$  be a Riemannian almost product manifold and  $\{e_i\}$  be a basis of the tangent space  $T_p M$  at a point  $p \in M$ . Let the components of the inverse matrix of  $g$  with respect to  $\{e_i\}$  be  $g^{ij}$ . If  $\rho$  and  $\tau$  are the Ricci tensor and the scalar curvature, then  $\rho^*$  and  $\tau^*$ , defined by  $\rho^*(y, z) = g^{ij} R(e_i, y, z, Pe_j)$  and  $\tau^* = g^{ij} \rho^*(e_i, e_j)$ , are called an *associated Ricci tensor* and an *associated scalar curvature*, respectively. We denote  $\tau^{**} = g^{ij} g^{ks} R(e_i, e_k, Pe_s, Pe_j)$ .

The *square norm* of  $\nabla P$  is defined by

$$\|\nabla P\|^2 = g^{ij} g^{ks} g((\nabla_{e_i} P) e_k, (\nabla_{e_j} P) e_s). \quad (7)$$

Obviously  $\|\nabla P\|^2 = 0$  iff  $(M, P, g)$  is a Riemannian  $P$ -manifold. In [6] it is proved that if  $(M, P, g)$  is a Riemannian  $\mathcal{W}_3$ -manifold then

$$\|\nabla P\|^2 = -2g^{ij} g^{ks} g((\nabla_{e_i} P) e_k, (\nabla_{e_s} P) e_j) = 2(\tau - \tau^{**}). \quad (8)$$

A tensor  $L$  of type (0, 4) with the properties

$$L(x, y, z, w) = -L(y, x, z, w) = -L(x, y, w, z), \quad (9)$$

$$\mathfrak{S}_{x, y, z} L(x, y, z, w) = 0 \quad (\text{the first Bianchi identity}) \quad (10)$$

is called a *curvature-like tensor*. Moreover, if the curvature-like tensor  $L$  has the property

$$L(x, y, Pz, Pw) = L(x, y, z, w), \quad (11)$$

we call it a *Riemannian  $P$ -tensor*.

If the curvature tensor  $R$  on a Riemannian  $\mathcal{W}_3$ -manifold  $(M, P, g)$  is a Riemannian  $P$ -tensor, i.e.  $R(x, y, Pz, Pw) = R(x, y, z, w)$ , then  $\tau^{**} = \tau$ . Therefore  $\|\nabla P\|^2 = 0$ , i.e.  $(M, P, g)$  is a Riemannian  $P$ -manifold.

### 3 Natural connection on Riemannian almost product manifolds

Let  $\nabla'$  be a linear connection with a tensor  $Q$  of the transformation  $\nabla \rightarrow \nabla'$  and a torsion tensor  $T$ , i.e.

$$\nabla'_x y = \nabla_x y + Q(x, y), \quad T(x, y) = \nabla'_x y - \nabla'_y x - [x, y].$$

The corresponding (0,3)-tensors are defined by

$$Q(x, y, z) = g(Q(x, y), z), \quad T(x, y, z) = g(T(x, y), z). \quad (12)$$

The symmetry of the Levi-Civita connection implies

$$T(x, y) = Q(x, y) - Q(y, x), \quad (13)$$

$$T(x, y) = -T(y, x). \quad (14)$$

A partial decomposition of the space  $\mathcal{T}$  of the torsion tensors  $T$  of type (0,3) (i.e.  $T(x, y, z) = -T(y, x, z)$ ) is valid on a Riemannian almost product manifold  $(M, P, g)$ :  $\mathcal{T} = \mathcal{T}_1 \oplus \mathcal{T}_2 \oplus \mathcal{T}_3 \oplus \mathcal{T}_4$ , where  $\mathcal{T}_i$  ( $i = 1, 2, 3, 4$ ) are invariant orthogonal subspaces [7]. For the projection operators  $p_i$  of  $\mathcal{T}$  in  $\mathcal{T}_i$  is established:

$$\begin{aligned} p_1(x, y, z) &= \frac{1}{8} \{ 2T(x, y, z) - T(y, z, x) - T(z, x, y) - T(Pz, x, Py) \\ &\quad + T(Py, z, Px) + T(z, Px, Py) - 2T(Px, Py, z) \\ &\quad + T(Py, Pz, x) + T(Pz, Px, y) - T(y, Pz, Px) \}, \\ p_2(x, y, z) &= \frac{1}{8} \{ 2T(x, y, z) + T(y, z, x) + T(z, x, y) + T(Pz, x, Py) \\ &\quad - T(Py, z, Px) - T(z, Px, Py) - 2T(Px, Py, z) \\ &\quad - T(Py, Pz, x) - T(Pz, Px, y) + T(y, Pz, Px) \}, \\ p_3(x, y, z) &= \frac{1}{4} \{ T(x, y, z) + T(Px, Py, z) - T(Px, y, Pz) \\ &\quad - T(x, Py, Pz) \}, \\ p_4(x, y, z) &= \frac{1}{4} \{ T(x, y, z) + T(Px, Py, z) + T(Px, y, Pz) \\ &\quad + T(x, Py, Pz) \}. \end{aligned}$$

A linear connection  $\nabla'$  on a Riemannian almost product manifold  $(M, P, g)$  is called a *natural connection* if  $\nabla'P = \nabla'g = 0$ . The last conditions are equivalent to  $\nabla'g = \nabla'\tilde{g} = 0$ . If  $\nabla'$  is a linear connection with a tensor  $Q$  of the transformation  $\nabla \rightarrow \nabla'$  on a Riemannian almost product manifold, then it is a natural connection iff the following conditions are valid:

$$F(x, y, z) = Q(x, y, Pz) - Q(x, Py, z), \quad (15)$$

$$Q(x, y, z) = -Q(x, z, y). \quad (16)$$

Let  $\Phi$  be the  $(0,3)$ -tensor determined by

$$\Phi(x, y, z) = g \left( \tilde{\nabla}_x y - \nabla_x y, z \right), \quad (17)$$

where  $\tilde{\nabla}$  is the Levi-Civita connection of the associated metric  $\tilde{g}$ .

**Theorem 3.1** ([7]). *A linear connection with the torsion tensor  $T$  on a Riemannian almost product manifold  $(M, P, g)$  is natural iff*

$$\begin{aligned} 4p_1(x, y, z) = & -\Phi(x, y, z) + \Phi(y, z, x) - \Phi(x, Py, Pz) \\ & - \Phi(y, Pz, Px) + 2\Phi(z, Px, Py), \end{aligned} \quad (18)$$

$$4p_3(x, y, z) = -g(N(x, y), z) = -2 \{ \Phi(z, Px, Py) + \Phi(z, x, y) \}. \quad (19)$$

In [9] it is proved that the both basic tensors  $F$  and  $\Phi$  on a Riemannian almost product manifold  $(M, P, g)$  are related as follows:

$$\Phi(x, y, z) = \frac{1}{2} \{ -F(Pz, x, y) + F(x, y, Pz) + F(y, Pz, x) \}, \quad (20)$$

$$F(x, y, z) = \Phi(x, y, Pz) + \Phi(x, z, Py). \quad (21)$$

If  $(M, P, g)$  is a Riemannian  $\mathcal{W}_3$ -manifold then (3), (4) and (20) imply

$$\Phi(x, y, z) = -F(x, Py, z) - F(y, Px, z), \quad (22)$$

which is equivalent to

$$\Phi(x, y, z) = -F(Pz, x, y). \quad (23)$$

**Theorem 3.2.** *For a natural connection with a torsion tensor  $T$  on a Riemannian  $\mathcal{W}_3$ -manifold  $(M, P, g)$ , which is not Riemannian  $P$ -manifold, the following properties are valid*

$$p_1 = 0, \quad p_3 \neq 0. \quad (24)$$

*Proof.* From equalities (23), (18), (3), (4) we get  $p_1 = 0$ . If we suppose  $p_3 = 0$  then (19) implies  $N = 0$ . Because of the last condition and  $N^* = 0$ , the manifold  $(M, P, g)$  becomes a Riemannian  $P$ -manifold, which is a contradiction. Therefore,  $p_3 \neq 0$  is valid.  $\square$

## 4 Canonical connection on Riemannian $\mathcal{W}_3$ -manifolds

**Definition 4.1** ([7]). *A natural connection with torsion tensor  $T$  on a Riemannian almost product manifold  $(M, P, g)$  is called a canonical connection if*

$$T(x, y, z) + T(y, z, x) + T(Px, y, Pz) + T(y, Pz, Px) = 0. \quad (25)$$

In [7] it is shown that (25) is equivalent to the condition

$$p_2 = p_4 = 0, \quad (26)$$

i.e. to the condition  $T \in \mathcal{T}_1 \oplus \mathcal{T}_3$ . The same paper shows that on every Riemannian almost product manifold  $(M, P, g)$  there exists a unique canonical connection  $\nabla'$ , and it is determined by

$$g(\nabla'_x y, z) = g(\nabla_x y, z) + \frac{1}{4} \{ \Phi(x, y, z) - 2\Phi(z, x, y) - \Phi(x, Py, Pz) \}. \quad (27)$$

For the torsion tensor  $T$  of this connection it is valid

$$T(x, y, z) = \frac{1}{4} \{ \Phi(y, z, x) - \Phi(z, x, y) + \Phi(y, Pz, Px) + \Phi(Pz, x, Py) \}. \quad (28)$$

By virtue of (28) and (23) we obtain the following property for a Riemannian  $\mathcal{W}_3$ -manifold

$$T(Px, y) = -PT(x, y). \quad (29)$$

Then the torsion tensor  $T$  of the canonical connection on a Riemannian  $\mathcal{W}_3$ -manifold has the properties:

$$T(x, y, z) = -T(y, x, z), \quad T(Px, y, z) = T(x, Py, z) = -T(x, y, Pz). \quad (30)$$

From Theorem 3.2 and condition (26) we obtain immediately the following

**Theorem 4.1.** *For the torsion tensor  $T$  of the canonical connection on a Riemannian  $\mathcal{W}_3$ -manifold the equality  $T = p_3$  is valid, i.e.  $T \in \mathcal{T}_3$ .  $\square$*

Equalities (23) and (27) imply the following

**Proposition 4.2.** *The canonical connection  $\nabla'$  on a Riemannian  $\mathcal{W}_3$ -manifold  $(M, P, g)$  is determined by*

$$\nabla'_x y = \nabla_x y + \frac{1}{4} \{ -(\nabla_y P) Px + (\nabla_{Py} P) x - 2(\nabla_x P) Py \}. \quad (31)$$

$\square$

Let  $\nabla'$  be the canonical connection on a Riemannian  $\mathcal{W}_3$ -manifold  $(M, P, g)$ . According to (31), for the tensor  $Q$  of the transformation  $\nabla \rightarrow \nabla'$  we have

$$Q(x, y) = \frac{1}{4} \{ -(\nabla_y P) Px + (\nabla_{Py} P) x - 2(\nabla_x P) Py \}. \quad (32)$$

Then

$$T(x, y) = -\frac{1}{2} \{ (\nabla_x P) Py + (\nabla_{Px} P) y \}.$$

Hence, having in mind  $N^* = 0$ , (2) and (12), we obtain

$$T(x, y, z) = -\frac{1}{2} \{ F(x, Py, z) + F(Px, y, z) \}. \quad (33)$$

Substituting  $y \leftrightarrow z$  into the above, according to (3), we get

$$T(x, z, y) = \frac{1}{2} \{ F(x, Py, z) - F(Px, y, z) \}.$$

Subtracting this from (33) and replacing  $y$  with  $Py$  in the result, we have

$$F(x, y, z) = T(x, z, Py) - T(x, Py, z). \quad (34)$$

The equalities (32), (12) and (2) imply

$$Q(x, y, z) = -\frac{1}{4} \{ F(y, Px, z) - F(Py, x, z) + 2F(x, Py, z) \}. \quad (35)$$

Hence, because of (3) and (4), we conclude that

$$Q(x, y, z) = -Q(y, x, z) - F(Pz, x, y). \quad (36)$$

**Theorem 4.3.** *Let  $\tau'$  and  $\tau$  be the scalar curvatures for the canonical connection  $\nabla'$  and the Levi-Civita connection  $\nabla$ , respectively, on a Riemannian  $\mathcal{W}_3$ -manifold. Then*

$$\tau' = \tau + \frac{1}{8} \|\nabla P\|^2. \quad (37)$$

*Proof.* According to (1) and (3), for a Riemannian almost product manifold we have  $g^{ij} F(Pz, e_i, e_j) = 0$ . Then, from (36), after contraction by  $x = e_i$ ,  $y = e_j$ , we obtain

$$g^{ij} Q(e_i, e_j, z) = 0. \quad (38)$$

Because of  $\nabla g^{ij} = 0$  (for the Levi-Civita connection  $\nabla$ ) and (38), we get

$$g^{ij} (\nabla_x Q) (e_i, e_j, z) = 0. \quad (39)$$

It is known that for the curvature tensors  $R'$  and  $R$  of  $\nabla'$  and  $\nabla$ , respectively, the following is valid:

$$\begin{aligned} R'(x, y, z, w) &= R(x, y, z, w) + (\nabla_x Q)(y, z, w) - (\nabla_y Q)(x, z, w) \\ &\quad + Q(x, Q(y, z), w) - Q(y, Q(x, z), w). \end{aligned}$$

Then from (16) and (12) it follows that

$$\begin{aligned} R'(x, y, z, w) &= R(x, y, z, w) + (\nabla_x Q)(y, z, w) - (\nabla_y Q)(x, z, w) \\ &\quad - g(Q(x, w), Q(y, z)) + g(Q(y, w), Q(x, z)), \end{aligned} \quad (40)$$

for a Riemannian almost product manifold  $(M, P, g)$ .

Using a contraction by  $x = e_i$ ,  $w = e_j$  in (40) and combining (16), (38) and (39), we find that the Ricci tensors  $\rho'$  and  $\rho$  for  $\nabla'$  and  $\nabla$  satisfy

$$\rho'(y, z) = \rho(y, z) + g^{ij} (\nabla_{e_i} Q)(y, z, e_j) + g^{ij} g(Q(y, e_j), Q(e_i, z)). \quad (41)$$

Similarly, after a contraction by  $y = e_k$ ,  $z = e_s$  in (41) and according to (39), we obtain

$$\tau' = \tau + g^{ij} g^{ks} g(Q(e_k, e_j), Q(e_i, e_s)), \quad (42)$$

for the scalar curvatures  $\tau'$  and  $\tau$  for  $\nabla'$  and  $\nabla$ . The equalities (42) and (32) imply

$$g^{ij} g^{ks} g(Q(e_k, e_j), Q(e_i, e_s)) = \frac{1}{16} g^{ij} g^{ks} g(A_{jk}, A_{si}) \quad (43)$$

for a Riemannian  $\mathcal{W}_3$ -manifold  $(M, P, g)$ , where

$$A_{jk} = -(\nabla_{e_j} P) P e_k + (\nabla_{P e_j} P) e_k - 2(\nabla_{e_k} P) P e_j.$$

From (43), (1), (7) and (8) we get

$$g^{ij} g^{ks} g(Q(e_k, e_j), Q(e_i, e_s)) = \frac{1}{8} \|\nabla P\|^2.$$

The last equality and (42) imply (37). □

**Corollary 4.4.** *A Riemannian  $\mathcal{W}_3$ -manifold is a Riemannian  $P$ -manifold if and only if the scalar curvatures for the canonical connection and the Levi-Civita connection are equal.* □



## 5 Canonical connection with Riemannian $P$ -tensor of curvature on a Riemannian $\mathcal{W}_3$ -manifold

The curvature tensor  $R'$  of a natural connection  $\nabla'$  on a Riemannian almost product manifold  $(M, P, g)$  satisfies property (9), according to (40). Since  $\nabla'P = 0$ , the property (11) is also valid. Therefore,  $R'$  is Riemannian  $P$ -tensor iff the first Bianchi identity (10) is satisfied. On the other hand, it is known ([3]) that for every linear connection  $\nabla'$  with a torsion  $T$  and a curvature tensor  $R'$  the following equality (the first Bianchi identity) is valid

$$\mathfrak{S}_{x,y,z} R'(x, y)z = \mathfrak{S}_{x,y,z} \{(\nabla'_x T)(y, z) + T(T(x, y), z)\}.$$

Since we have  $\nabla'g = 0$ , the last equality implies

$$\mathfrak{S}_{x,y,z} R'(x, y, z, w) = \mathfrak{S}_{x,y,z} \{(\nabla'_x T)(y, z, w) + T(T(x, y), z, w)\}.$$

Thus,  $R'$  satisfies (10) iff

$$\mathfrak{S}_{x,y,z} \{(\nabla'_x T)(y, z, w) + T(T(x, y), z, w)\} = 0. \quad (44)$$

This leads to the following

**Lemma 5.1.** *The curvature tensor for the natural connection  $\nabla'$  with a torsion  $T$  on a Riemannian almost product manifold is a Riemannian  $P$ -tensor iff (44) is valid.  $\square$*

We substitute  $Pz$  for  $z$  and  $Pw$  for  $w$  in (44). Hence, according to (30), we obtain

$$\begin{aligned} & (\nabla'_x T)(y, z, w) - (\nabla'_y T)(z, x, w) + (\nabla'_{Pz} T)(x, y, Pw) \\ & + T(T(x, y), z, w) + T(T(y, Pz), x, w) + T(T(Pz, x), y, Pw) = 0. \end{aligned}$$

We add the last equality to (44), and substitute  $Px$  for  $x$  and  $Pw$  for  $w$  in the result. Then, using (30), we get

$$\begin{aligned} & (\nabla'_z T)(x, y, z) - (\nabla'_{Pz} T)(x, Py, w) \\ & + 2T(T(y, z), x, w) + 2T(T(z, x), y, w) = 0. \end{aligned} \quad (45)$$

We substitute  $Py, Pz$  for  $y, z$ , respectively, and we apply (30). We subtract the obtained equality from (26) and we reapply (30). This leads to

$$T(T(z, x), y, w) = 0.$$

Hence, (34) and (3) imply

$$F(Py, w, T(z, x)) = -T(y, w, T(z, x)),$$

and from (2) and (12) we obtain

$$g(T(x, z), T(y, w) - (\nabla_{Py}P)w) = 0. \quad (46)$$

Since, according to (33) and (2), we have

$$T(y, w) = -\frac{1}{2}\{(\nabla_yP)Pw + (\nabla_{Py}P)w\},$$

the following equality is valid

$$T(y, w) + (\nabla_{Py}P)w = -\frac{1}{2}\{(\nabla_yP)Pw - (\nabla_{Py}P)w\}.$$

Thus, using (46), we arrive at the following

**Theorem 5.2.** *Let  $(M, P, g)$  be a Riemannian  $\mathcal{W}_3$ -manifold, whose canonical connection has a Riemannian  $P$ -tensor of curvature. Then the following equality is valid*

$$g((\nabla_xP)Pz + (\nabla_{Px}P)z, (\nabla_{Py}P)w - (\nabla_yP)Pw) = 0.$$

□

## 6 Canonical connection with parallel torsion on a Riemannian $\mathcal{W}_3$ -manifold

In this section we consider a canonical connection  $\nabla'$  with parallel torsion  $T$  (i.e.  $\nabla'T = 0$ ) on a Riemannian  $\mathcal{W}_3$ -manifold  $(M, P, g)$ .

According to the Hayden theorem ([2])

$$Q(x, y, z) = \frac{1}{2}\{T(x, y, z) - T(y, z, x) + T(z, x, y)\}. \quad (47)$$

Combining this with (13), (15), (35), leads to the following

**Proposition 6.1.** *Let  $\nabla'$  be a natural connection on a Riemannian almost product manifold  $(M, P, g)$ . Then the tensors  $T$ ,  $Q$  and  $F$  are parallel or non-parallel at the same time with respect to  $\nabla'$ .* □

Let  $\nabla'$  be a natural connection with parallel torsion on a Riemannian almost product manifold  $(M, P, g)$ . According to Proposition 6.1 we have  $\nabla'Q = 0$ . Then, having in mind the formula for the covariant derivative of  $Q$ , we obtain

$$xQ(y, z, w) - Q(\nabla'_x y, z, w) - Q(y, \nabla'_x z, w) - Q(y, z, \nabla'_x w) = 0. \quad (48)$$

Since  $Q$  is the tensor of the deformation  $\nabla \rightarrow \nabla'$ , applying the formula for the covariant derivative of  $Q$  with respect to  $\nabla$  and equalities (12) and (13), we obtain the following

**Lemma 6.2.** *Let  $R'$  be the curvature tensor for a natural connection  $\nabla'$  with a parallel torsion  $T$  on a Riemannian almost product manifold  $(M, P, g)$ . Then the following equality is valid*

$$\begin{aligned} R'(x, y, z, w) &= R(x, y, z, w) + Q(T(x, y), z, w) \\ &\quad + g(Q(y, z), Q(x, w)) - g(Q(x, z), Q(y, w)). \end{aligned} \quad (49)$$

□

Let  $(M, P, g)$  be a Riemannian  $\mathcal{W}_3$ -manifold whose canonical connection  $\nabla'$  has a parallel torsion  $T$ . Then, according to (16), (36) and (2), we have

$$Q(T(x, y), z, w) = g(Q(z, w), T(x, y)) - g((\nabla_{Pw} P)z, T(x, y)).$$

The last equality and Lemma 6.2 imply

**Theorem 6.3.** *Let  $(M, P, g)$  be a Riemannian  $\mathcal{W}_3$ -manifold whose canonical connection  $\nabla'$  has a parallel torsion  $T$ . Then for the curvature tensor  $R'$  of  $\nabla'$  we obtain*

$$\begin{aligned} R'(x, y, z, w) &= R(x, y, z, w) \\ &\quad + g(Q(y, z), Q(x, w)) - g(Q(x, z), Q(y, w)) \\ &\quad + g(Q(z, w), T(x, y)) - g((\nabla_{Pw} P)z, T(x, y)). \end{aligned} \quad (50)$$

□

Because of (38) we have  $g^{ij}Q(e_i, e_j) = 0$ . Then, from (50) via a contraction by  $x = e_i$ ,  $w = e_j$ , we get

$$\begin{aligned} \rho'(y, z) &= \rho(y, z) - g^{ij}g(Q(e_i, z), Q(y, e_j)) \\ &\quad + g^{ij}g(Q(z, e_j), T(e_i, y)) - g^{ij}g((\nabla_{Pe_j} P)z, T(e_i, y)), \end{aligned} \quad (51)$$

where  $\rho'$  and  $\rho$  are the Ricci tensors for  $\nabla'$  and  $\nabla$ , respectively.

Combining (12), (36), (30), (4), (13), (3) and (2), we obtain

$$\begin{aligned} g(Q(z, e_j), T(e_i, y)) &= g(Q(e_j, z), Q(y, e_i)) - g(Q(e_j, z), Q(e_i, y)) \\ &\quad + g((\nabla_{Pe_j} P) z, T(e_i, y)) + g((\nabla_{Pz} P) e_j, T(e_i, y)). \end{aligned} \quad (52)$$

We get the following equality from (51) and (52):

$$\rho'(y, z) = \rho(y, z) - g^{ij} g(Q(e_j, z), Q(e_i, y)) + g^{ij} g((\nabla_{Pz} P) e_j, T(e_i, y)). \quad (53)$$

A contraction by  $y = e_k, z = e_s$  leads to

$$\tau' = \tau - g^{ij} g^{ks} g(Q(e_j, e_s), Q(e_i, e_k)) + g^{ij} g^{ks} g((\nabla_{Pe_s} P) e_j, T(e_i, e_k)), \quad (54)$$

where  $\tau'$  and  $\tau$  are the respective scalar curvatures for  $\nabla'$  and  $\nabla$ .

Using (31), (8) and (7), we get

$$g^{ij} g^{ks} g(Q(e_j, e_s), Q(e_i, e_k)) = \frac{1}{4} \|\nabla P\|^2. \quad (55)$$

From (8) and  $2T(e_i, e_j) = -(\nabla_{e_i} P) P e_k + (\nabla_{Pe_j} P) e_k$  we have

$$g^{ij} g^{ks} g((\nabla_{Pe_s} P) e_j, T(e_i, e_k)) = \frac{1}{2} \|\nabla P\|^2. \quad (56)$$

Then, (54), (55) and (56) imply

$$\tau' = \tau + \frac{1}{4} \|\nabla P\|^2.$$

From the last equality and (37) we obtain the following

**Theorem 6.4.** *Let  $(M, P, g)$  be a Riemannian  $\mathcal{W}_3$ -manifold whose canonical connection  $\nabla'$  has a parallel torsion  $T$ . Then  $(M, P, g)$  is Riemannian  $P$ -manifold.  $\square$*

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*Dimitar Mekerov*  
*University of Plovdiv*  
*Faculty of Mathematics and Informatics*  
*Department of Geometry*  
*236 Bulgaria blvd.*  
*Plovdiv 4003*  
*Bulgaria*  
*e-mail: mircho@uni-plovdiv.bg*